Want to be an environmental scientist? Better be ready to get your hands dirty!

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There are all kinds of talking birds. The common crow is able to repeat a few words, while a bird called a Bedgerigar (or common parakeet) can have a vocabulary of thousands of words. A bird of this type named Puck was found in 1995 to have a vocabulary of 1728 words.

African Grey Parrots are also remarkable, not only for their knowledge of words, but also for their other mental abilities. In 2005, an African Grey Parrot named Alex was reported to have understood the concept of zero!
Many Terms

Previously, you worked with a number of expressions in the form $ax + b$ and $ax^2 + bx + c$. Each of these is also part of a larger group of expressions known as polynomials.

A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form $ax^k$, where $a$ is any real number and $k$ is a non-negative integer. In general, a polynomial is of the form $a_1x^{k_1} + a_2x^{k_2} + \ldots + a_nx^{k_n}$. Within a polynomial, each product is a term, and the number being multiplied by a power is a coefficient.

The polynomial $m^3 + 8m^2 - 10m + 5$ has four terms. Each term is written in the form $ax^k$.

<table>
<thead>
<tr>
<th>Term</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Power</td>
<td>$m^3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exponent</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Write each term from the worked example and then identify the coefficient, power, and exponent. The first term has already been completed for you.

2. Analyze each polynomial. Identify the terms and coefficients in each.
   a. $-2x^2 + 100x$
   b. $x^2 + 4x + 3$
   c. $4m^3 - 2m^2 + 5$
Polynomials are named according to the number of terms they have. Polynomials with only one term are ***monomials***. Polynomials with exactly two terms are ***binomials***. Polynomials with exactly three terms are ***trinomials***.

The **degree of a term** in a polynomial is the exponent of the term. The greatest exponent in a polynomial determines the **degree of the polynomial**. In the polynomial $4x + 3$, the greatest exponent is 1, so the degree of the polynomial is 1.

3. Khalil says that $3x^2 + 4x - 1$ is a polynomial with a degree of 1 because 1 is the greatest exponent and it is a trinomial because it has 3 terms. Jazmin disagrees and says that this is not a polynomial at all because the power on the first term is not a whole number. Who is correct? Explain your reasoning.

4. Describe why each expression is not a polynomial.

   a. $\frac{4}{x}$

   b. $\sqrt{x}$
5. Cut out each polynomial.
Identify the degree of each polynomial and then analyze and sort according to the number of terms of the polynomial. Finally, glue the sorted polynomials in the appropriate column of the table.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4x - 6x^2$</td>
<td></td>
</tr>
<tr>
<td>$125p$</td>
<td></td>
</tr>
<tr>
<td>$\frac{4}{5}r^3 + \frac{2}{5}r - 1$</td>
<td></td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td></td>
</tr>
<tr>
<td>$y^2 - 4y + 10$</td>
<td></td>
</tr>
<tr>
<td>$5 - 7h$</td>
<td></td>
</tr>
<tr>
<td>$-3 + 7n + n^2$</td>
<td></td>
</tr>
<tr>
<td>$-6$</td>
<td></td>
</tr>
<tr>
<td>$-13s + 6$</td>
<td></td>
</tr>
<tr>
<td>$12.5t^3$</td>
<td></td>
</tr>
<tr>
<td>$78j^3 - 3j$</td>
<td></td>
</tr>
<tr>
<td>$25 - 18m^2$</td>
<td></td>
</tr>
</tbody>
</table>
A polynomial is written in standard form when the terms are in descending order, starting with the term with the greatest degree and ending with the term with the least degree.

6. Analyze the polynomials you just sorted. Identify the polynomials not written in standard form and write them in standard form.
PROBLEM 2 Something to Squawk About

You are playing a new virtual reality game called “Species.” You are an environmental scientist who is responsible for tracking two species of endangered parrots, the Orange-bellied Parrot and the Yellow-headed Parrot. Suppose the Orange-bellied Parrots’ population can be modeled by the function

\[ B(x) = -18x + 120, \]

where \( x \) represents the number of years since the current year. Then suppose that the population of the Yellow-headed Parrot can be modeled by the function

\[ H(x) = 4x^2 - 5x + 25. \]

The graphs of the two polynomial functions are shown.

Your new task in the game is to determine the total number of these endangered parrots over a six-year span. You can calculate the total population of parrots using the two graphed functions.

1. Use the graphs of \( B(x) \) and \( H(x) \) to determine the function, \( T(x) \), to represent the total population of parrots.
   a. Write \( T(x) \) in terms of \( B(x) \) and \( H(x) \).

   b. Predict the shape of the graph of \( T(x) \).

   c. Sketch a graph of \( T(x) \) on the coordinate plane shown. First choose any 5 \( x \)-values and add their corresponding \( y \)-values to create a new point on the graph of \( T(x) \). Then connect the points with a smooth curve.

   d. Did your sketch match your prediction in part (b)? Describe the function family \( T(x) \) belongs to.
2. Use a graphing calculator to check your sketch by graphing $B(x)$ and $H(x)$ and then the sum of the two functions.
   a. In $Y_1$, enter the function for $B(x)$.
   b. In $Y_2$, enter the function for $H(x)$.
   c. In $Y_3$, enter the sum of the functions $B(x)$ and $H(x)$.
      (Since $Y_1 = B(x)$ and $Y_2 = H(x)$, you can enter $Y_1 + Y_2$ in $Y_3$ to represent the sum of $B(x)$ and $H(x)$.)

3. Complete the table.

<table>
<thead>
<tr>
<th>Time Since Present (years)</th>
<th>Number of Orange-bellied Parrots</th>
<th>Number of Yellow-headed Parrots</th>
<th>Total Number of Parrots</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. How did you calculate the total number of parrots?

5. Write a function, $T(x)$, in terms of $x$ that can be used to calculate the total number of parrots at any time.
6. Analyze the function you wrote in Question 5.
   a. Identify the like terms of the polynomial functions.

   b. Use the Associative Property to group the like terms together.

   c. Combine like terms to simplify the expression.

   d. Graph the function you wrote in part (c) to verify that it matches the graph of the sum of the two functions in Y3.

7. Use your new polynomial function to confirm that your solution in the table for each time is correct.
   a. 3 years ago
   b. currently
   c. 3 years from now

8. Zoe says that using T(x) will not work for any time after 6 years from now because by that point the Orange-bellied Parrot will be extinct. Is Zoe’s statement correct? Why or why not?
Throughout the game “Species,” you must always keep track of the difference between the populations of each type of species. If the difference gets to be too great, you lose the game.

9. Use the graphs of $B(x)$ and $H(x)$ to determine the function, $D(x)$, to represent the difference between the populations of each type of species.
   a. Write $D(x)$ in terms of $B(x)$ and $H(x)$.
   b. Predict the shape of the graph of $D(x)$.
   c. Sketch a graph of $D(x)$ on the coordinate plane in Question 1. Choose any 5 $x$-values and subtract their corresponding $y$-values to create a new point to form the graph of $D(x)$. Then connect the points with a smooth curve.
   d. Did your sketch match your prediction in part (b)? Describe the function family $D(x)$ belongs to.

10. Use a graphing calculator to check your sketch by graphing $B(x)$ and $H(x)$ and then the difference of the two functions.
    a. For $Y_1$, enter the function for $B(x)$.
    b. For $Y_2$, enter the function for $H(x)$.
    c. For $Y_3$, enter the difference of the functions $B(x)$ and $H(x)$.
       (Since $Y_1 = B(x)$ and $Y_2 = H(x)$, you can enter $Y_1 - Y_2$ in $Y_3$ to represent $B(x) - H(x)$.)
11. Consider the original functions, $B(x) = -18x + 120$ and $H(x) = 4x^2 - 5x + 25$.

   a. Write a new function, $D(x)$, in terms of $x$ to represent the difference between the population of Orange-bellied Parrots and the population of Yellow-headed Parrots.

   b. Because you are subtracting one function from another function, you must subtract each term of the second function from the first function. To do this, use the Distributive Property to distribute the negative sign to each term in the second function.

   c. Now, combine the like terms to simplify the expression.

   d. Graph the function you wrote in part (c) to verify that it matches the graph of the difference of the two functions in $Y_3$.

12. For each time given, use the table in Question 2 to determine the difference between the population of Orange-bellied Parrots and the population of Yellow-headed Parrots. Then, use the function you wrote in Question 8, part (c) to verify your solutions.

   a. 3 years ago  
   b. currently  
   c. 3 years from now
13. Eric uses his function $D(x) = -4x^2 - 13x + 95$, to determine that the difference between the number of Orange-bellied Parrots and the number of Yellow-headed Parrots 7 years from now will be 192. Is Eric correct or incorrect? If he is correct, explain to him what his answer means in terms of the problem situation. If he is incorrect, explain where he made his error and how to correct it.

$D(x) = -4(7)^2 - 13(7) + 95$
$= -196 - 91 + 95$
$D(x) = -192$

PROBLEM 3 Just the Math

1. Analyze the work. Determine the error and make the necessary corrections.

a. Marco

$3x^2 + 5x^2 = 8x^4$

b. Kamiah

$2x - (4x + 5)$
$2x - 4x + 5$
$-2x + 5$
2. Determine the sum or difference of each. Show your work.
   a. \((x^2 - 2x - 3) + (2x + 1)\)  
   b. \((2x^2 + 3x - 4) - (2x^2 + 5x - 6)\)

   c. \((4x^3 + 5x - 2) + (7x^2 - 8x + 9)\)  
   d. \((9x^4 - 5) - (8x^4 - 2x^3 + x)\)
KenKen has been a popular mathematics puzzle game around the world since at least 2004. The goal is to fill in the board with the digits 1 to whatever, depending on the size of the board. If it's a $5 \times 5$ board, only the digits 1 through 5 can be used. If it's a $6 \times 6$ board, only the digits 1 through 6 can be used. Each row and column must contain the numbers 1 through whatever without repeating numbers.

Many KenKen puzzles have regions called “cages” outlined by dark bold lines. In each cage, you must determine a certain number of digits that satisfy the rule. For example, in the cage “$2 \div$” shown, you have to determine two digits that divide to result in 2.

Can you solve this KenKen?
So far, you have learned how to add and subtract polynomials. But what about multiplying polynomials?

Let’s consider the binomials $x + 1$ and $x + 2$. You can use algebra tiles to model the two binomials and determine their product.

Represent each binomial with algebra tiles.

\[
\begin{align*}
x + 1 & \\
x + 2 &
\end{align*}
\]

Create an area model using each binomial.

\[
\begin{array}{c}
\times \\
x + 1 \\
x + 2 \\
1 \times \\
1 \\
\end{array}
\]

1. What is the product of $(x + 1)(x + 2)$?

2. How would the model change if the binomial $x + 2$ was changed to $x + 4$. What is the new product of $x + 1$ and $x + 4$?
Jamaal represented the product of \((x + 1)\) and \((x + 2)\) as shown.

Natalie looked at the area model and told Jamaal that he incorrectly represented the area model because it does not look like the model in the example. Jamaal replied that it doesn’t matter how the binomials are arranged in the model.

Determine who’s correct and use mathematical principles or properties to support your answer.
4. Use algebra tiles to determine the product of the binomials in each.

a. $x + 2$ and $x + 3$

b. $x + 2$ and $x + 4$

c. $2x + 3$ and $3x + 1$

You can use a graphing calculator to check if the product of two binomials is correct.

**Step 1:** Press $Y=$. Enter the two binomials multiplied next to $Y_1$. Then enter their product next to $Y_2$.

To distinguish between the graphs of $Y_1$ and $Y_2$, move your cursor to the left of $Y_2$ until the $\backslash$ flashes. Press ENTER one time to select the bold $\backslash$.

**Step 2:** Press WINDOW to set the bounds and intervals for the graph.

**Step 3:** Press GRAPH.
5. Use a graphing calculator to verify the product from the worked example:
\((x + 1)(x + 2) = x^2 + 3x + 2\).
   a. Sketch both graphs on the coordinate plane.

   ![Graph of \((x + 1)(x + 2)\) and \(x^2 + 3x + 2\)]

   b. How do the graphs verify that \((x + 1)(x + 2)\) and \(x^2 + 3x + 2\) are equivalent?

   c. Plot and label the \(x\)-intercepts and the \(y\)-intercept on your graph. How do the forms of each expression help you identify these points?

6. Verify that the products you determined in Question 5, part (a) through part (c) are correct using your graphing calculator. Write each pair of factors and the product. Then sketch each graph on the coordinate plane.
   a. \(x^2 + 2x + 2\)
Recall that \( r_1 \) and \( r_2 \) are the \( x \)-intercepts of a function written in factored form, 
\[
f(x) = a(x - r_1)(x - r_2),
\]
where \( a \neq 0 \).

7. How can you determine whether the products in Question 5, part (a) through part (c) are correct using factored form? Explain your reasoning.
I'm Running Out of Algebra Tiles!

While using algebra tiles is one way to determine the product of polynomials, they can also become difficult to use when the terms of the polynomials become more complex.

Todd was calculating the product of the binomials $4x + 7$ and $5x - 3$. He thought he didn’t have enough algebra tiles to determine the product. Instead, he performed the calculation using the model shown.

1. Describe how Todd calculated the product of $4x + 7$ and $5x - 3$.

2. How is Todd's method similar to and different from using the algebra tiles method?

Todd used a multiplication table to calculate the product of the two binomials. By using a multiplication table, you can organize the terms of the binomials as factors of multiplication expressions. You can then use the Distributive Property of Multiplication to multiply each term of the first polynomial with each term of the second polynomial.

Recall the problem Making the Most of the Ghosts in Chapter 11. In it, you wrote the function $r(x) = (50 - x)(100 + 10x)$, where the first binomial represented the possible price reduction of a ghost tour, and the second binomial represented the number of tours booked if the price decrease was $x$ dollars per tour.

3. Determine the product of $(50 - x)$ and $(100 + 10x)$ using a multiplication table.
4. Determine the product of the binomials using multiplication tables. Write the product in standard form.
   a. $3u + 17$ and $4u - 6$
   b. $8x + 16$ and $6x + 3$
   c. $7y - 14$ and $8y - 4$
   d. $9y - 4$ and $y + 5$

5. Describe the degree of the product when you multiply two binomials with a degree of 1.
PROBLEM 3  You Have Been Distributing the Whole Time!

So far, you have used both algebra tiles and multiplication tables to determine the product of two polynomials.

Let’s look at the original area model and think about multiplying a different way. The factors and equivalent product for this model are:

\[(x + 1)(x + 2) = x^2 + 3x + 2\]

The model can also be shown as the sum of each row.

1. Write the factors and the equivalent product for each row represented in the model.

2. Use your answers to Question 1 to rewrite \( (x + 1)(x + 2) \).
   a. Complete the first equivalent statement using the factors from each row.
   b. Next, write an equivalent statement using the products of each row.

\[(x + 1)(x + 2) = \underline{\underline{\phantom{x^2}}} \underline{\underline{\phantom{x^2}}} \underline{\underline{\phantom{3x}}} \underline{\underline{\phantom{2}}}\]

\[= \underline{\underline{\phantom{x^2}}} \underline{\underline{\phantom{x^2}}} \underline{\underline{\phantom{3x}}} \underline{\underline{\phantom{2}}}\]

\[= x^2 + 3x + 2\]

   c. Write the justification for each step.
To multiply the polynomials \( x + 5 \) and \( x - 2 \), you can use the Distributive Property. 

First, use the Distributive Property to multiply each term of \( x + 5 \) by the entire binomial \( x - 2 \).

\[ (x + 5)(x - 2) = (x)(x - 2) + (5)(x - 2) \]

Now, distribute \( x \) to each term of \( x - 2 \) and distribute 5 to each term of \( x - 2 \).

\[ x^2 - 2x + 5x - 10 \]

Finally, collect the like terms and write the solution in standard form.

\[ x^2 + 3x - 10 \]
Another method that can be used to multiply polynomials is called the FOIL method. The word FOIL indicates the order in which you multiply the terms. You multiply the First terms, then the Outer Terms, then the Inner terms, and then the Last terms. FOIL stands for First, Outer, Inner, Last.

You can use the FOIL method to determine the product of \((x + 1)\) and \((x + 2)\).

<table>
<thead>
<tr>
<th>First</th>
<th>Outer</th>
<th>Inner</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x + 1)(x + 2) = x^2)</td>
<td>((x + 1)(x + 2) = 2x)</td>
<td>((x + 1)(x + 2) = x)</td>
<td>((x + 1)(x + 2) = 2)</td>
</tr>
</tbody>
</table>

Collect the like terms and write the solution in standard form.

\[ x^2 + 3x + 2 \]

5. Determine each product.
   
a. \(2x(x + 3)\)
   
b. \(5x(7x - 1)\)
   
c. \((x + 1)(x + 3)\)
   
d. \((x - 4)(2x + 3)\)
1. Consider the polynomials $x + 1$ and $x^2 - 3x + 2$. You need to use the Distributive Property twice to determine the product.

First, use the Distributive Property to multiply each term of $x + 1$ by the polynomial $x^2 - 3x + 2$.

$$(x + 1)(x^2 - 3x + 2) = (x)(x^2 - 3x + 2) + (1)(x^2 - 3x + 2)$$

Now, distribute $x$ to each term of $x^2 - 3x + 2$, and distribute $1$ to each term of $x^2 - 3x + 2$.

$$(x + 1)(x^2 - 3x + 2) = (x)(x^2) + (x)(-3x) + (x)(2) + (1)(x^2) + (1)(-3x) + (1)(2)$$

Finally, multiply and collect the like terms and write the solution in standard form.

\[x^3 - 3x^2 + 2x + x^2 - 3x + 2\]

\[x^3 - 2x^2 - x + 2\]
3. You can also use a multiplication table to multiply a binomial by a trinomial. Complete the table to determine the product.

<table>
<thead>
<tr>
<th></th>
<th>$x^2$</th>
<th>$-3x$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Did you get the same product as the worked example shows?

4. Determine each product.
   a. $(x - 5)(x^2 + 3x + 1)$
   b. $(x + 5)(2x^2 - 3x - 4)$

Using multiplication tables may help you stay organized.
c. \((x - 4)(x^2 - 8x + 16)\)

How can you use your graphing calculator to verify that your products are correct?

Be prepared to share your solutions and methods.
The history of the word “multiply” is an interesting one. You probably know that “multi-” means “many.” But did you know that the rest of the word, based on the Latin “plicare,” means “to fold”?

This is why you might read in older texts that someone increased their money “twofold” or “tenfold.”

Multiplication is closely related to folding. Can you see how?
**PROBLEM 1 What About the Other Way Around?**

In the previous lesson, you multiplied polynomials. More specifically, you multiplied two linear expressions to determine a quadratic expression. In this lesson, you will go in reverse and think about how to take a polynomial represented as the sum of terms and write an equivalent expression in factored form, if it is possible. To factor an expression means to rewrite the expression as a product of factors.

One way to factor an expression is to factor out the greatest common factor first.

Consider the polynomial $3x + 15$.

The greatest common factor is 3.

$$3x + 15 = 3x + 3(5)$$
$$= 3(x + 5)$$

Therefore, $3x + 15 = 3(x + 5)$.

In order to factor out a greatest common factor, use the Distributive Property in reverse. Recall that, using the Distributive Property, you can rewrite the expression $ab + ac$ as $a(b + c)$.

1. Factor out the greatest common factor for each polynomial, if possible.
   a. $4x + 12$
   b. $x^3 - 5x$
   c. $3x^2 - 9x - 3$
   d. $-x - 7$
   e. $2x - 11$
   f. $5x^2 - 10x + 5$
2. How can you check to see if you factored out the greatest common factor of each correctly?

**PROBLEM 2  Factoring Trinomials**

In the previous chapter, you used a graphing calculator to rewrite a quadratic expression in factored form, $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

Now, let's consider a strategy for factoring quadratic expressions without the use of technology. Understanding that the product of two linear expressions produces a quadratic expression is necessary for understanding how to factor a quadratic expression.

Remember, factoring is the reverse of multiplying.
An area model can be used to factor $x^2 + 7x + 6$.

First, represent each part of the trinomial as a piece of the area model. In this problem, $x^2 + 7x + 6$ consists of one $x^2$ block, seven $x$ blocks, and six constant blocks.

Second, use all of the pieces to form a rectangle. In this problem, the parts can only be arranged in one way.

1. Write the trinomial as the product of the two factors.
   
   $x^2 + 7x + 6 = $

2. Factor each trinomial using an area model.
   
   a. $x^2 + 5x + 4 = $

   Do you remember the idea of zero pairs when you learned to subtract integers? So, if you need to add any tiles to your model to create the rectangle, you need to make sure you don’t change the value of the original trinomial. Hint: if you add an “$x$” tile, you must also add a “$-x$” tile!
b. $x^2 - 6x + 9 =$

c. $x^2 + 5x - 6 =$

3. Look back at the quadratic trinomials you just factored using area models. What do you notice about the constant term of the trinomial, $c$, and the constants of the binomial factors $r_1$ and $r_2$?
Factor the trinomial \(x^2 + 10x + 16\).

Start by writing the leading term \((x^2)\) and the constant term \(16\) in the table.

<table>
<thead>
<tr>
<th>(\cdot)</th>
<th>(\cdot)</th>
<th>(x^2)</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(x)</td>
<td>(x^2)</td>
<td>16</td>
</tr>
</tbody>
</table>

Factor the leading term.

<table>
<thead>
<tr>
<th>(\cdot)</th>
<th>(x)</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(x^2)</td>
<td>8x</td>
</tr>
<tr>
<td>2</td>
<td>2x</td>
<td>16</td>
</tr>
</tbody>
</table>

The sum of \(2x\) and \(8x\) is \(10x\).

So, \(x^2 + 10x + 16 = (x + 2)(x + 8)\).

4. Explain why the other factor pairs for \(c = 16\) do not work.
   a. \((1)(16)\)
   b. \((4)(4)\)

5. Use multiplication tables to factor each trinomial.
   a. \(x^2 + 9x + 20\)
Another method for factoring a trinomial is trial and error, using the factors of the leading term, \( ax^2 \), and the constant term, \( c \).

Factor the trinomial \( x^2 + 11x + 18 \).

To factor the quadratic expression, you must consider the factors of \( ax^2 \) and \( c \).

Factors of the leading term, \( x^2 \), are \( x \) and \( x \).

Factors of the constant term, \(-24\), are:\n\(-1, 24 \) and \(1, -24\)\n\(-2, 12 \) and \(2, -12\)\n\(-3, 8 \) and \(3, -8\)\n\(-4, 6 \) and \(4, -6\).

Use these factor pairs to write binomial products. Determine which binomials produce the correct middle term.

\[
\begin{align*}
(x - 1)(x + 24) &= x^2 + 23x - 24 & \text{not correct} \\
(x + 1)(x - 24) &= x^2 - 23x - 24 & \text{not correct} \\
(x - 2)(x + 12) &= x^2 + 10x - 24 & \text{not correct} \\
(x + 2)(x - 12) &= x^2 - 10x - 24 & \text{correct} \\
(x - 3)(x + 8) &= x^2 + 5x - 24 & \text{not correct} \\
(x + 3)(x - 8) &= x^2 - 5x - 24 & \text{not correct} \\
(x - 4)(x + 6) &= x^2 + 2x - 24 & \text{not correct} \\
(x + 4)(x - 6) &= x^2 - 2x - 24 & \text{not correct}
\end{align*}
\]

Always look for a greatest common factor first. Notice that in this trinomial the GCF is 1.
6. Factor each trinomial using the method from the worked example. List the factor pairs.
   a. $x^2 + 5x - 24$
   b. $x^2 - 3x - 28$

7. Consider the two examples shown.

   a. Compare the two given trinomials. What is the same and what is different about the $a$, $b$, and $c$ values?
   b. Compare the factored form of each trinomial. What do you notice?
8. Choose from the list to write the correct factored form for each trinomial.

a. \( x^2 + 5x + 4 = \) ____________
   - \((x + 1)(x - 4)\)
   - \((x + 1)(x + 4)\)
   - \((x - 1)(x + 4)\)
   - \((x - 1)(x - 4)\)

b. \( 2x^2 + 7x + 3 = \) ____________
   - \((2x - 1)(x - 3)\)
   - \((2x - 1)(x + 3)\)
   - \((2x + 1)(x + 3)\)
   - \((2x + 1)(x - 3)\)

c. \( x^2 + 7x + 10 = \) ____________
   - \((x - 2)(x + 5)\)
   - \((x + 2)(x + 5)\)
   - \((x - 2)(x - 5)\)
   - \((x + 2)(x - 5)\)

9. Analyze the signs of each quadratic expression written in standard form and the operations in the binomial factors in Question 8. Then complete each sentence.

<table>
<thead>
<tr>
<th>the same</th>
<th>both positive</th>
<th>one positive and one negative</th>
<th>different</th>
<th>both negative</th>
</tr>
</thead>
</table>

a. If the constant term is positive, then the operations in the binomial factors are ____________.

b. If the constant term is positive and the middle term is positive, then the operations in the binomial factors are ____________.

c. If the constant term is positive and the middle term is negative, then the operations in the binomial factors are ____________.

d. If the constant term is negative, then the operations in the binomial factors are ____________.

e. If the constant term is negative and the middle term is positive, then the operations in the binomial factors are ____________.

f. If the constant term is negative and the middle term is negative, then the operations in the binomial factors are ____________.
10. Factor each quadratic expression.
   a. \( x^2 + 8x + 15 = \) 
      \( x^2 - 8x + 15 = \) 
      \( x^2 + 2x - 15 = \) 
      \( x^2 - 2x - 15 = \) 
   b. \( x^2 + 10x + 24 = \) 
      \( x^2 - 10x + 24 = \) 
      \( x^2 + 2x - 24 = \) 
      \( x^2 - 2x - 24 = \) 

11. Elaine, Maggie, and Grace were asked to factor the trinomial \( 15 + 2x - x^2 \).

   Grace
   \( 15 + 2x - x^2 \)
   \( (5 - x)(3 + x) \)

   Elaine
   \( 15 + 2x - x^2 \)
   \( (5 - x)(3 + x) \)
   \( (x - 5)(x + 3) \)

   Maggie
   \( 15 + 2x - x^2 \)
   \( -x^2 + 2x + 15 \)
   \( -(x^2 - 2x - 15) \)
   \( -(x - 5)(x + 3) \)

Who’s correct? Determine which student is correct and explain how that student determined the factored form. If a student is not correct, state why and make the correction.
12. Marilyn and Jake were working together to factor the trinomial $4x^2 + 22x + 24$. They first noticed that there was a greatest common factor and rewrote the trinomial as $2(2x^2 + 11x + 12)$.

Next, they considered the factor pairs for $2x^2$ and the factor pairs for 12.

$2x^2$: (2x) (x)
12: (1) (12)
     (2) (6)
     (3) (4)

Marilyn listed all out all the possible combinations.

$2(2x + 1)(x + 12)$
$2(2x + 12)(x + 1)$
$2(2x + 2)(x + 6)$
$2(2x + 6)(x + 2)$
$2(2x + 3)(x + 4)$
$2(2x + 4)(x + 3)$

Jake immediately eliminated four out of the six possible combinations because the terms of one of the linear expressions contained common factors.

$2(2x + 1)(x + 12)$
$2(2x + 12)(x + 1)$
$2(2x + 2)(x + 6)$
$2(2x + 6)(x + 2)$
$2(2x + 3)(x + 4)$
$2(2x + 4)(x + 3)$

Explain Jake’s reasoning. Then circle the correct factored form of $4x^2 + 22x + 24$. 
Talk the Talk

1. Factor each polynomial completely. First, determine if there is a greatest common factor, and then write the polynomial in factored form.
   a. \( x^2 - 9x - 10 \)  
   b. \( 4x^2 - 20x + 16 \)
   
   c. \( -20 + 9b - b^2 \)  
   d. \( 3y^2 - 8y - 3 \)
   
   e. \( 7x^2 - 7x - 56 \)  
   f. \( 3y^3 - 27y^2 - 30y \)

2. Use the word bank to complete each sentence. Then explain your reasoning.
   
   always  sometimes  never

   a. The product of two linear expressions will ______________ be a trinomial with a degree of 3.

   b. The two binomial factors of a quadratic expression will ______________ have a degree of one.

   c. The factoring of a quadratic expression will ______________ result in two binomials.

Be prepared to share your solutions and methods.
Zeroing In

Solving Quadratics by Factoring

**LEARNING GOALS**

In this lesson, you will:

- Solve quadratic equations and functions using factoring.
- Connect the zeros of a function to the $x$-intercepts of a graph.
- Determine the roots of quadratic equations.

**KEY TERMS**

- Zero Product Property
- Converse of Multiplication Property of Zero
- roots

The word zero has had a long and interesting history. The word comes from the Hindu word *sunya*, which meant “void” or “emptiness.” In Arabic, this word became *sifr*, which is also where the word *cipher* comes from. In Latin, it was changed to *cephirum*, and finally, in Italian it became *zevero* or *zefiro*, which was shortened to *zero*.

The ancient Greeks, who were responsible for creating much of modern formal mathematics, did not even believe zero was a number!
PROBLEM 1  Roots of Quadratic Equations

Recall that a quadratic expression of the form \( x^2 + bx + c \) can be factored using an area model, a multiplication table, or the factors of the constant term \( c \). The quadratic expression \( x^2 - 4x - 5 \) is factored using each method as shown.

- **Area model**

\[
\begin{array}{c|cccc}
& x & -1 & -1 & -1 \\
\hline
x^2 & -x & -x & -x & -x \\
1 & -1 & -1 & -1 & -1 \\
\end{array}
\]

\[ x^2 - 4x - 5 = (x - 5)(x + 1) \]

- **Multiplication table**

| \(
\begin{array}{c|c|c}
\cdot & x & -5 \\
\hline
x & x^2 & -5x \\
1 & x & -5 \\
\end{array}
\) |

\[ x^2 - 4x - 5 = (x - 5)(x + 1) \]

- **Factors of the constant term \( c \)**

Factors of \( -5 \): \(-5, 1 \quad -1, 5 \)

Sums: \(-5 + 1 = -4 \quad 5 + (-1) = 4 \)

\[ x^2 - 4x - 5 = (x - 5)(x + 1) \]

The **Zero Product Property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

If \( ab = 0 \), then \( a = 0 \) or \( b = 0 \).

This is also referred to as the **Converse of the Multiplication Property of Zero**.

1. Use the Zero Product Property to determine the solutions of the quadratic equation \( x^2 - 4x - 5 = 0 \). Then, check your solutions by substituting back into the original equation.
2. Let’s examine the quadratic equation \( 0 = x^2 - 4x - 5 \).

a. Graph both sides of the quadratic equation on the coordinate plane shown.

b. Rewrite the equation in factored form.

c. Identify the vertex, \( x \)- and \( y \)-intercepts, and the axis of symmetry.
   - \( y \)-intercept:
   - \( x \)-intercept(s):
   - axis of symmetry:
   - vertex:

![Graph of the quadratic equation](image)

d. Compare the \( x \)-intercepts of the equation \( y = x^2 - 4x - 5 \) to the solutions to Question 1. What do you notice?

e. Compare the intersections of the two equations you graphed to the solutions to Question 1. What do you notice?
The solutions to a quadratic equation are called roots. The roots indicate where the graph of a quadratic equation crosses the x-axis. So, roots, zeros, and x-intercepts are all related.

To calculate the roots of a quadratic equation using factoring:

- Perform transformations so that one side of the equation is equal to zero.
- Factor the quadratic expression on the other side of the equation.
- Set each factor equal to zero.
- Solve the resulting equations for the roots. Check each solution in the original equation.

You can calculate the roots for the quadratic equation \( x^2 - 4x = -3 \).

\[
\begin{align*}
  x^2 - 4x &= -3 \\
  x^2 - 4x + 3 &= -3 + 3 \\
  x^2 - 4x + 3 &= 0 \\
  (x - 3)(x - 1) &= 0 \\
  (x - 3) &= 0 \quad \text{or} \quad (x - 1) = 0 \\
  x - 3 + 3 &= 0 + 3 \quad \text{or} \quad x - 1 + 1 = 0 + 1 \\
  x &= 3 \quad \text{or} \quad x = 1
\end{align*}
\]

Check:

\[
\begin{align*}
  x^2 - 4x &= (3)^2 - 4(3) = 9 - 12 = -3 \\
  x^2 - 4x &= (1)^2 - 4(1) = 1 - 4 = -3
\end{align*}
\]

Determine the roots of each quadratic equation.

3. \( x^2 - 8x + 12 = 0 \)

4. \( x^2 - 5x - 24 = 0 \)
5. $x^2 + 10x - 75 = 0$

6. $x^2 - 11x = 0$

7. $x^2 + 8x = -7$

8. $x^2 - 5x = 13x - 81$
9. \(3x^2 - 22x + 7 = 0\)

10. \(8x^2 + 2x - 21 = 0\)
Calculate the zeros of each quadratic function, or the roots of each quadratic equation, if possible.

1. \( f(x) = x^2 - 7x - 18 \)

2. \( f(x) = x^2 - 11x + 12 \)

3. \( f(x) = x^2 + 10x - 39 \)

4. \( 2x^2 + 4x = 0 \)

5. \( \frac{2}{3}x^2 - \frac{5}{6}x = 0 \)

Be prepared to share your solutions and methods.
What Makes You So Special?

Special Products

LEARNING GOALS

In this lesson, you will:

• Identify and factor the difference of two squares.
• Identify and factor perfect square trinomials.
• Solve quadratic equations and functions using factoring.
• Identify and factor the difference of two cubes.
• Identify and factor the sum of cubes.

KEY TERMS

• difference of two squares
• perfect square trinomial
• difference of two cubes
• sum of two cubes

There are a number of rare elements on Earth. Precious gems are relatively rare, which is why they’re so valuable.

Some blood types are rare too. The O blood type is the most common, with about 37% of the population having it. The least common is the blood type AB, with only about 4% of the population having it.

People with rare blood types are strongly encouraged to donate blood if they can, since it is more difficult to find rare blood types in cases of emergency.

What is your blood type?
1. Multiply the binomials.
   a. \((x - 4)(x + 4) = \) ____________________  
      \((x + 4)(x + 4) = \) ____________________  
      \((x - 4)(x - 4) = \) ____________________  
   b. \((x - 3)(x + 3) = \) ____________________  
      \((x + 3)(x + 3) = \) ____________________  
      \((x - 3)(x - 3) = \) ____________________  
   c. \((3x - 1)(3x + 1) = \) ____________________  
      \((3x + 1)(3x + 1) = \) ____________________  
      \((3x - 1)(3x - 1) = \) ____________________  
   d. \((2x - 1)(2x + 1) = \) ____________________  
      \((2x + 1)(2x + 1) = \) ____________________  
      \((2x - 1)(2x - 1) = \) ____________________  

2. What patterns do you notice between the factors and the products?

3. Multiply these binomials.
   \((ax - b)(ax + b) = \) ____________________  
   \((ax + b)(ax + b) = \) ____________________  
   \((ax - b)(ax - b) = \) ____________________

In Questions 1 and 3, you should have observed a few special products. The first type of special product is called the difference of two squares. The difference of two squares is an expression in the form \(a^2 - b^2\) that can be factored as \((a + b)(a - b)\).

The second type of special product is called a perfect square trinomial. A perfect square trinomial is an expression in the form \(a^2 + 2ab + b^2\) or in the form \(a^2 - 2ab + b^2\). A perfect square trinomial can be written as the square of a binomial.

4. Identify the expressions in Questions 1 and 3 that are examples of the difference of two squares. Write both the unfactored and factored forms of each expression.
5. Identify the expressions in Questions 1 and 3 that are examples of perfect square trinomials. Write both the unfactored and factored forms of each expression.
   a. Of the form $ax^2 + 2ab + b^2$:  
   b. Of the form $ax^2 - 2ab + b^2$:  

6. Tizeh says that he can factor the sum of two squares in the same way as he factors the difference of two squares. It's just that addition will be used in both binomials. His work is shown.

   Tizeh
   $$x^2 + 16 = (x + 4)(x + 4)$$

Cheyanne disagrees and says that to factor the sum of two squares, you should use subtraction in each binomial. Her work is shown.

   Cheyanne
   $$x^2 + 16 = (x - 4)(x - 4)$$

Who is correct? Explain your reasoning.
7. Complete the table to represent each difference of squares.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( a^2 - b^2 )</th>
<th>Factored Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>2</td>
<td>( 4x^2 - 9 )</td>
<td>((x^2 + 4)(x^2 - 4))</td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( y^2 )</td>
<td></td>
<td></td>
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</tbody>
</table>

a. What does the information in this table show?

b. Can any of the expressions in factored form be factored further? If so, factor them further.

8. Factor each polynomial, if possible.

   a. \( x^2 + 10x + 25 \)
   b. \( 4x^2 + 20x + 25 \)
   c. \( x^2 - 24x + 144 \)
   d. \( 36x^2 - 36x + 9 \)
   e. \( x^2 + 25 \)
   f. \( 16x^4 - 1 \)

9. Calculate the roots of each quadratic equation.
   a. \( x^2 - 12x + 36 = 0 \)
   b. \( 9x^2 - 25 = 0 \)
10. Calculate the zeros of each function.
   a. \( f(x) = 25x^2 + 20x + 4 \)
   b. \( f(x) = 9x^2 + 1 \)
   c. \( f(x) = 9 - 24x + 16x^2 \)
   d. \( f(x) = \frac{1}{4}x^2 - 1 \)

**PROBLEM 2 Are Cubes Perfect Too?**

In Problem 1, you dealt with special products that had degrees of 2. There are also special products with degrees of 3.

1. Use a multiplication table to determine \((x - 2)(x^2 + 2x + 4)\).

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2. Use a multiplication table to determine \((x - y)(x^2 + xy + y^2)\).

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3. Analyze your solution in Question 2.
   a. What happened to the original terms \( x \) and \( y \)?

   b. Does the solution in Question 2 follow the same pattern as the solution in Question 1? Explain your reasoning.

Each expression and product in Questions 1 and 2 represents the **difference of two cubes**. The **difference of two cubes** is an expression in the form \( a^3 - b^3 \) that can be factored as \((a - b)(a^2 + ab + b^2)\).

4. Each part of the factored form is related to the cube root of each term in the original expression. Identify each part of the factored form as it relates to the cube root of one of the original terms.

   \[
   a^3 - b^3 = (a - b)(a^2 + ab + b^2)
   \]

5. Using the parts you just identified, explain the formula for the difference of two cubes.

6. Factor the difference of the two cubes: \( x^3 - 27 \).
Let's consider the products of two more polynomials in factored form.

7. Use multiplication tables to determine each product.
   a. \((x + 5)(x^2 - 5x + 25)\)

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   b. \((x + y)(x^2 - xy + y^2)\)

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Each product represents the sum of two cubes. Previously you determined that you cannot factor the sum of two squares. Based on your products in Question 7, you can factor the sum of two cubes. The sum of two cubes is an expression in the form \(a^3 + b^3\) that can be factored as \((a + b)(a^2 - ab + b^2)\).

8. Factor the sum of the two cubes: \(u^3 + 8\).
9. Analyze the expression Emilio factored.

Emilio

\[64x^3 + 125 = (4x)^3 + 5^3\]
\[= (4x + 5)(4x^2 - 4x(5) + 5^2)\]
\[= (4x + 5)(4x^2 - 20x + 25)\]

Explain to Emilio what he did wrong and correctly write the expression in factored form.

10. Sophie factored the expression shown.

Sophie

\[250x^4 + 128x^2 = 2(25x^2 + 64)\]
\[= 2(5x + 4)(5x^2 - (5x)(4) + 4^2)\]
\[= 2(5x + 4)(25x^2 - 20x + 16)\]

Explain Sophie’s mistake and correctly write the expression in factored form.

11. Completely factor the expression \(x^6 - y^6\).
1. Complete the table to define each special product.

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Factors</th>
<th>Definition</th>
<th>Example</th>
</tr>
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<tbody>
<tr>
<td>Perfect Square Trinomial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difference of squares</td>
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<tr>
<td>Sum of Cubes</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Difference of Cubes</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

2. Factor each expression.
   a. \(2x^2 + 18\)                    b. \(9x^6 - y^8\)
   
   c. \(2x^3 - 16\)                    d. \(125a^3 + 27\)

Be prepared to share your solutions and methods.
One of the most brilliant of ancient Greek mathematicians was a man named Pythagoras. He believed that every number could be expressed as a ratio of two integers.

Yet, legend tells us that one day at sea, one of Pythagoras’s students pointed out to him that the diagonal of a square which measures 1 unit by 1 unit would be $\sqrt{2}$, a number which could not possibly be represented as a ratio of two integers.

This student was allegedly thrown overboard and the rest of the group was sworn to secrecy!
Good Vibrations

Vanessa plays guitar and knows that when she plucks a guitar string, the vibrations result in sound waves, which are the compression and expansion of air particles. Each compression and expansion is called a cycle. The number of cycles that occur in one second is 1 hertz. The number of cycles that occur in one second can be referred to as “wave speed,” or “frequency.”

Vanessa also knows that tuning a guitar requires changing the tension of the strings. The tension can be thought of as the amount of stretch (in pounds of pressure per inch) on a string between two fixed points. A string with the correct tension produces the correct number of cycles per second over time, which produces the correct tone.

1. Consider a guitar string that has a tension of 0.0026 pound per inch. An equation that relates hertz \( h \) and tension \( t \) in pounds per inch is \( h^2 = t \).
   a. Determine the string tension if the frequency is 9.5 hertz.
   b. Determine the string tension if the frequency is 7.6 hertz.
   c. Determine the string tension if the frequency is 8.5 hertz.

2. What appears to happen to the tension as the frequency increases?

3. Write an equation to determine the frequency produced by a string with a tension of 81 pounds per inch. Use the same variables you were given in Question 1.

4. Write an equation to determine the frequency produced by a string with a tension of 36 pounds per inch. Use the same variables you were given in Question 1.
Notice that your answers to Questions 3 and 4 are square roots of 81 and 36. A number $b$ is a square root of $a$ if $b^2 = a$. So, 9 is the square root of 81 because $9^2 = 81$, and 6 is a square root of 36 because $6^2 = 36$.

5. Jasmine claims that 81 could have two square roots: 9 and $-9$. Maria says that there can be only one square root for 81, which is 9. Determine who is correct and explain why that student is correct.

In earlier grades, you may have seen problems in which you only determined one square root. However, there are in fact two square roots for every whole number, a positive square root (which is also called the principal square root) and a negative square root. This occurs because of the rule you learned about multiplying two negative numbers: when two negative numbers are multiplied together, the product is a positive number.

To solve the equation

$$h^2 = 81,$$

you can extract the square root from both sides of the equation.

$$\sqrt{h^2} = \pm \sqrt{81}$$

$$h = \pm 9$$

However, you must still be mindful of the solutions in the context of the problem.

6. Lincoln determined the frequency, in hertz, for a string with a tension of 121 pounds per inch. His work is shown.

Lincoln

- $h^2 = 121$
- $h = \pm \sqrt{121}$
- $h^2 = \pm 11$

The frequency must be 11 hertz because the square of 11 is 121.

Explain why Lincoln is correct in the context of the problem.
Generally, experts agree that humans can hear frequencies that are between 15 hertz and 20,000 hertz.

Recall the equation $h^2 = t$. The A-string on a guitar has a frequency of 440 hertz (cycles per second).

1. Determine the string tension if the frequency of the A-string is 440 hertz.

If $440^2 = 193,600$, then $\sqrt{193,600} = 440$. This second expression is a **radical expression** because it involves a radical symbol ($\sqrt{}$). The **radicand** is the expression enclosed within the radical symbol. In the expression $\sqrt{193,600}$, 193,600 is the radicand.

2. Write an equation to determine the frequency of a string with a tension of 75 pounds per inch.

3. Write your equation as a radical expression.

4. Can you predict whether the frequency will be a positive integer? Explain why or why not.

Remember, an integer is a member of the set of whole numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. 
You can also estimate the square roots of numbers that are not perfect squares.

You can determine the approximate value of $\sqrt{75}$.

Determine the perfect square that is closest to but less than 75.
Then determine the perfect square that is closest to but greater than 75.

$64 \leq 75 \leq 81$

Determine the square roots of the perfect squares.

$\sqrt{64} = 8 \quad \sqrt{75} = ? \quad \sqrt{81} = 9$

Now that you know that $\sqrt{75}$ is between 8 and 9, you can test the squares of numbers between 8 and 9.

$8.6^2 = 73.96 \quad 8.7^2 = 75.69$

Since 75.69 is closer to 75 than 73.96, 8.7 is the approximate square root of $\sqrt{75}$.

5. Determine the approximate frequency for each string tension given. First, write a quadratic equation with the information given, and then approximate your solution to the nearest tenth.
   a. 42 pounds per inch

   b. 50 pounds per inch
13. 136 pounds per inch

6. The lowest key on a piano has a string tension of 756 pounds of pressure per inch. What is the frequency of the lowest key on the piano? Write an equation and approximate your answer to the nearest tenth.

PROBLEM 3  No! It Must Be Exact

There are times when an exact solution is necessary. For example, acoustical engineers may need to calculate the exact solution when designing sound stages or studios.

Laura makes and designs acoustical tiles for recording studios. These tiles are used to reflect different instrument tones of different frequencies in the studio. One of her clients has requested that the area of each square acoustical tile needs to be 75 square inches. Because these tiles can be lined up in rows or columns and they affect other factors in a recording studio, Laura needs to determine the exact side measure of each acoustical tile.
Consider a square acoustical tile with an area of 75 square inches.

You can set up and solve an equation to determine exact side length of the square.

\[ s^2 = 75 \]
\[ s = \sqrt{75} \]

You can also rewrite \( \sqrt{75} \) in a different form to help show the value. First, rewrite the product of 75 to include any perfect square factors, and then extract the square roots of those perfect squares.

\[ \sqrt{75} = \sqrt{25 \cdot 3} \]
\[ = \sqrt{25} \cdot \sqrt{3} \]
\[ = 5 \cdot \sqrt{3} \]

The exact measure of each side of the acoustical tile is \( \sqrt{75} \), or 5\( \sqrt{3} \) inches.

1. Estimate the value of 5\( \sqrt{3} \). Explain your reasoning.

2. Compare your approximation of 5\( \sqrt{3} \) to the approximation of \( \sqrt{75} \) from the worked example in Problem 2. What do you notice?
3. Rewrite each radical by extracting all perfect squares, if possible.
   a. \( \sqrt{20} \)
   b. \( \sqrt{26} \)
   c. \( \sqrt{64} \)

4. For each given area, write an equation to determine the side measurements of the square acoustical tiles. Then, determine the exact side measurement of each square acoustical tile.
   a. 18 square inches
   b. 116 square inches
Can extracting the square root also be used for expressions containing a variable and a constant term that are squared?

Consider the equation \((x - 1)^2 = 17\). How can you determine the value of \(x\)?

First take the square root of both sides of the equation.

\[
(x - 1)^2 = 17
\]

\[
\sqrt{(x - 1)^2} = \pm\sqrt{17}
\]

\[
x - 1 = \pm\sqrt{17}
\]

Then, isolate the variable to determine the value of the variable.

\[
x - 1 + 1 = \pm\sqrt{17} + 1
\]

\[
x = 1 \pm \sqrt{17}
\]

\[
x \approx 1 \pm 4.12
\]

\[
x \approx -3.12 \text{ or } 5.12
\]

5. Determine the approximate solutions for each of the given equations.

a. \((r + 8)^2 = 83\)

b. \((17 - d)^2 = 55\)
6. Rewrite each radical by extracting all perfect squares, if possible.

   a. $\sqrt{18}$

   b. $\sqrt{116}$

   c. $5\sqrt{24}$

   d. $7\sqrt{99}$
Another Method
Completing the Square

LEARNING GOALS
In this lesson, you will:
• Determine the roots of a quadratic equation by completing the square.
• Complete the square geometrically and algebraically.

KEY TERMS
• completing the square

Can you construct a square that has the exact same area as a given circle using only a compass and a straightedge?

Well, no. And this was proven to be impossible in 1882, making pi a “transcendental” irrational number.

Unfortunately, it seems that no one in Indiana got the message at the time. In 1897, the Indiana state legislature, via amateur mathematician Edwin Goodwin, tried to pass a law declaring that there was a solution to this famous problem—known as completing the square.
1. Factor each quadratic function and determine the zeros, if possible.
   a. \( f(x) = x^2 + 5x + 4 \)          b. \( f(x) = x^2 - 4x + 2 \)
   c. \( f(x) = x^2 + 5x + 2 \)          d. \( f(x) = x^2 - 4x - 5 \)

2. Were you able to determine the zeros of each function in Question 1 by factoring? Explain your reasoning.
3. If you cannot factor a quadratic function, does that mean it does not have zeros?
   a. Graph the quadratic function from Question 1, part (b) on your calculator.
      Sketch the graph on the coordinate plane.

   b. Does this function have zeros? Explain your reasoning.

   The quadratic function you graphed has zeros but cannot be factored, so we must find another method for calculating its zeros. You can use your understanding of the relationship among the coefficients of a perfect square trinomial to construct a procedure to solve any quadratic equation.

   **PROBLEM 2 Seeing the Square**

   Previously, you factored trinomials of the form $a^2 + 2ab + b^2$ as the perfect square $(a + b)^2$. This knowledge can help you when constructing a procedure for solving any quadratic equation.

   1. The expression $x^2 + 10x$ can be represented geometrically as shown. Write the area of each piece in the center of the piece.
2. This figure can now be modified into the shape of a square by splitting the second rectangle in half and rearranging the pieces.
   a. Label the side length of each piece and write the area of each piece in the center of the piece.

   b. Do the two figures represent the same expression? Explain your reasoning.

   c. Complete the figure so it is a square. Label the area of the piece you added.

   d. Add this area to your original expression. What is the new expression?

   e. Factor this expression.

The process you worked through above is a method known as completing the square. **Completing the square** is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

3. Use a geometric figure to complete the square for each expression. Then factor the resulting trinomial.
   a. \( x^2 + 8x \)
   b. \( x^2 + 5x \)
4. Analyze your work in Question 3.
   a. Explain how to complete the square on an expression $x^2 + bx$ where $b$ is an integer.

   b. Describe how the coefficient of the middle term, $b$, is related to the constant term, $c$ in each trinomial you wrote in Question 3.

5. Use the descriptions you provided in Question 4 to determine the unknown value of $b$ or $c$ that would make each expression a perfect square trinomial. Then write the expression as a binomial squared.

   a. $x^2 - 8x + _____ = _____$

   b. $x^2 + 5x + _____ = _____$

   c. $x^2 - _____ + 100 = _____$

   d. $x^2 + _____ + 144 = _____$
So how does completing the square help when trying to determine the roots of a quadratic equation that cannot be factored? Let’s take a look.

Determine the roots of the equation \( x^2 - 4x + 2 = 0 \).

Isolate \( x^2 - 4x \). You can make this into a perfect square trinomial.

\[
x^2 - 4x + 2 - 2 = 0 - 2
\]

\[
x^2 - 4x = -2
\]

Determine the constant term that would complete the square.

\[
x^2 - 4x + ___ = -2 + ___
\]

Add this term to both sides of the equation.

\[
x^2 - 4x + 4 = -2 + 4
\]

Factor the left side of the equation.

\[
(x - 2)^2 = 2
\]

Determine the square root of \( (x - 2)^2 \) on each side of the equation.

\[
\sqrt{(x - 2)^2} = \pm\sqrt{2}
\]

\[
x - 2 = \pm\sqrt{2}
\]

Set the factor of the perfect square trinomial equal to each of the square roots of the constant.

\[
x - 2 = \sqrt{2} \quad \text{or} \quad x - 2 = -\sqrt{2}
\]

\[
x = 2 + \sqrt{2} \quad \text{or} \quad x = 2 - \sqrt{2}
\]

Solve for \( x \).

\[
x \approx 3.414 \quad \text{or} \quad x \approx 0.5858
\]

The roots are approximately 3.41 and 0.59.

6. Check the solutions in the worked example by substituting each into the original equation.

7. Do your solutions match the zeros you sketched on your graph in Problem 1, Question 3? Explain how you determined your answer.
8. Determine the roots of each equation by completing the square.
   
   a. \(x^2 - 6x + 4 = 0\)

   b. \(x^2 - 12x + 6 = 0\)

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

\[y = ax^2 + bx + c\]

**Step 1:**

\[y - c = ax^2 + bx\]

**Step 2:**

\[y - c = a\left(x^2 + \frac{b}{a}x\right)\]

**Step 3:**

\[y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)\]

**Step 4:**

\[y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2\]

**Step 5:**

\[y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2\]

**Step 6:**

\[y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)\]

Notice that the a-value was factored out before completing the square!
9. Explain why $\frac{b}{2a}$ was added to the left side of the equation in Step 3.

10. Given a quadratic function in the form $y = ax^2 + bx + c$,
   a. identify the axis of symmetry.
   b. identify the location of the vertex.

11. Rewrite each quadratic equation in vertex form.
    Then identify the axis of symmetry and the location of the vertex in each.
   a. $y = x^2 + 8x - 9$
   b. $y = 3x^2 + 2x - 1$

12. Determine the roots and the location of the vertex of $y = x^2 + 20x + 36 = 0$.

Be prepared to share your solutions and methods.
Chapter 13 Summary

KEY TERMS

- polynomial (13.1)
- term (13.1)
- coefficient (13.1)
- monomial (13.1)
- binomial (13.1)
- trinomial (13.1)
- degree of a term (13.1)
- degree of a polynomial (13.1)
- Zero Product Property (13.4)
- Converse of Multiplication Property of Zero (13.4)
- roots (13.4)
- difference of two squares (13.5)
- perfect square trinomial (13.5)
- difference of two cubes (13.5)
- sum of two cubes (13.5)
- square root (13.6)
- positive square root (13.6)
- principal square root (13.6)
- negative square root (13.6)
- extract the square root (13.6)
- radical expression (13.6)
- radicand (13.6)
- completing the square (13.7)

13.1 Identifying Characteristics of Polynomial Expressions

A polynomial is an expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form \( ax^k \) where \( a \), called the coefficient, is a real number and \( k \) is a non-negative integer. In general, a polynomial is of the form \( a_1x^k + a_2x^{k-1} + \ldots + a_nx^0 \). Each of the products in a polynomial is called a term. Polynomials are named according to the number of terms: monomials have exactly 1 term, binomials have exactly 2 terms, and trinomials have exactly 3 terms. The exponent of a term is the degree of the term, and the greatest exponent in a polynomial is the degree of the polynomial. When a polynomial is written in standard form, the terms are written in descending order, with the term of the greatest degree first and ending with the term of the least degree.

Example

The characteristics of the polynomial \( 13x^3 + 5x + 9 \) are shown.

<table>
<thead>
<tr>
<th></th>
<th>1st term</th>
<th>2nd term</th>
<th>3rd term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>( 13x^3 )</td>
<td>( 5x )</td>
<td>( 9 )</td>
</tr>
<tr>
<td>Coefficient</td>
<td>( 13 )</td>
<td>( 5 )</td>
<td>( 9 )</td>
</tr>
<tr>
<td>Power</td>
<td>( x^3 )</td>
<td>( x^1 )</td>
<td>( x^0 )</td>
</tr>
<tr>
<td>Exponent</td>
<td>( 3 )</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

There are 3 terms in this polynomial. Therefore, this polynomial is a trinomial. This trinomial has a degree of 3 because 3 is the greatest degree of the terms in the trinomial.
13.1 Adding and Subtracting Polynomial Expressions

Polynomials can be added or subtracted by identifying the like terms of the polynomial functions, using the Associative Property to group the like terms together, and combining the like terms to simplify the expression.

Example 1

Expression: \((7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)\)

The like terms are \(7x^2\) and \(2x^2\) and \(-2x\) and \(-3x\). The terms \(8x^3\) and 12 are not like terms.

\((7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)\)

\(8x^3 + (7x^2 + 2x^2) + (-2x - 3x) + 12\)

\(8x^3 + 9x^2 - 5x + 12\)

Example 2

Expression: \((4x^4 + 7x^2 - 3) - (2x^2 - 5)\)

The like terms are \(7x^2\) and \(2x^2\) and \(-3\) and \(-5\). The term \(4x^4\) does not have a like term.

\((4x^4 + 7x^2 - 3) - (2x^2 - 5)\)

\(4x^4 + (7x^2 - 2x^2) + (-3 + 5)\)

\(4x^4 + 5x^2 + 2\)

13.2 Modeling the Product of Polynomials

The product of 2 binomials can be determined by using an area model with algebra tiles. Another way to model the product of 2 binomials is a multiplication table which organizes the two terms of the binomials as factors of multiplication expressions.

Example 1

\((2x + 1)(x + 3)\)

\((2x + 1)(x + 3) = 2x^2 + 7x + 3\)
Example 2

\((9x - 1)(5x + 7)\)

\[\begin{array}{ccc}
\cdot & 9x & -1 \\
5x & 45x^2 & -5x \\
7 & 63x & -7 \\
\end{array}\]

\((9x - 1)(5x + 7) = 45x^2 - 5x + 63x - 7 = 45x^2 + 58x - 7\)

### 13.2 Using the Distributive Property to Multiply Polynomials

The Distributive Property can be used to multiply polynomials. Depending on the number of terms in the polynomials, the Distributive Property may need to be used multiple times.

**Example**

\((2x^2 + 5x - 10)(x + 7)\)

\[(2x^2 + 5x - 10)(x) + (2x^2 + 5x - 10)(7)\]

\[(2x^2)(x) + (5x)(x) -10(x) + (2x^2)(7) + (5x)(7) - 10(7)\]

\[2x^3 + 5x^2 - 10x + 14x^2 + 35x - 70\]

\[2x^3 + 19x^2 + 25x - 70\)

### 13.3 Factoring Polynomials by Determining the Greatest Common Factor

Factoring a polynomial means to rewrite the expression as a product of factors. The first step in factoring any polynomial expression is to determine whether or not the expression has a greatest common factor.

**Example**

Expression: \(12x^3 + 4x^2 + 16x\)

The greatest common factor is \(4x\).

\[12x^3 + 4x^2 + 16x = 4x(3x^2) + 4x(x) + 4x(4) = 4x(3x^2 + x + 4)\]
13.3 Factoring Trinomials

A quadratic expression can be written in factored form, \( ax^2 + bx + c = a(x - r_1)(x - r_2) \), by using an area model with algebra tiles, multiplication tables, or trial and error. Factoring a quadratic expression means to rewrite it as a product of two linear expressions.

Example 1

Trinomial: \( x^2 + 3x + 2 \)

Represent each part of the trinomial as a piece of the area model. Then use the parts to form a rectangle.

![Diagram of area model]

The factors of this trinomial are the length and width of the rectangle. Therefore, \( x^2 + 3x + 2 = (x + 1)(x + 2) \).

Example 2

Trinomial: \( x^2 + 15x + 54 \)

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( x^2 )</td>
<td>( 9x )</td>
</tr>
<tr>
<td>6</td>
<td>( 6x )</td>
<td>54</td>
</tr>
</tbody>
</table>

So, \( x^2 + 15x + 54 = (x + 6)(x + 9) \).

13.4 Solving Quadratic Equations Using Factoring

The Zero Product Property states that if the product of two or more factors is equal to 0, at least one factor must be equal to 0. The property is also known as the Converse of the Multiplication Property of Zero. This property can be used to solve a quadratic equation. The solutions to a quadratic equation are called roots. To calculate the roots of a quadratic equation using factoring:

- Perform transformations so that one side of the equation is equal to 0.
- Factor the quadratic expression on the other side of the equation.
- Set each factor equal to 0.
- Solve the resulting equation for the roots. Check each solution in the original equation.
Connecting the Zeros of a Function to the $x$-intercepts of a Graph

The $x$-intercepts of the graph of the quadratic function $f(x) = ax^2 + bx + c$ and the zeros of the function are the same as the roots of the equation $ax^2 + bx + c = 0$.

Example
Function: $f(x) = x^2 + 6x - 55$

$x^2 + 6x - 55 = 0$

$(x + 11)(x - 5) = 0$

$x + 11 = 0$ or $x - 5 = 0$

$x = -11$ or $x = 5$

The zeros of the function $f(x) = x^2 + 6x - 55$ are $x = -11$ and $x = 5$.

Identifying Special Products of Degree 2

There are special products of degree 2 that have certain characteristics. A perfect square trinomial is a trinomial formed by multiplying a binomial by itself. A perfect square trinomial is in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. A binomial is a difference of two squares if it is in the form $a^2 - b^2$ and can be factored as $(a + b)(a - b)$.

Example 1
Expression: $4x^2 + 12x + 9$

$4x^2 + 12x + 9$

$(2x + 3)(2x + 3)$

$(2x + 3)^2$

This expression is a perfect square trinomial.
Example 2

Expression: $x^2 - 49y^2$

$x^2 - 49y^2$

$x^2 - (7y)^2$

$(x + 7y)(x - 7y)$

This binomial is the difference of two squares.

13.5 Identifying Special Products of Degree 3

There are also special products of degree 3 that have certain characteristics. The difference of two cubes is an expression in the form $a^3 - b^3$ that can be factored as $(a - b)(a^2 + ab + b^2)$. The sum of two cubes is an expression in the form of $a^3 + b^3$ that can be factored as $(a + b)(a^2 - ab + b^2)$.

Example 1

Expression: $27x^3 - 125$

$27x^3 - 125$

$(3x)^3 - 5^3$

$(3x - 5)((3x)^2 + (3x)(5) + 5^2)$

$(3x - 5)(9x^2 + 15x + 25)$

Example 2

Expression: $64x^3 + 1$

$64x^3 + 1$

$(4x)^3 + 1^3$

$(4x + 1)((4x)^2 - (4x)(1) + 1^2)$

$(4x + 1)(16x^2 - 4x + 1)$
Determining Approximate Square Roots of Given Values

A number \( b \) is a square root of \( a \) if \( b^2 = a \). There are 2 square roots for every whole number: a positive square root, which is also called the principal square root, and a negative square root. To determine approximate square roots for given values, determine the perfect square that is closest to, but less than, the given value. Also, determine the perfect square that is closest to, but greater than, the given value. You can use these square roots to approximate the square root of the given number.

**Example**

Determine the approximate value of \( \sqrt{40} \).

\[
36 \leq 40 \leq 49
\]

\[
\sqrt{36} = 6 \quad \sqrt{40} = ? \quad \sqrt{49} = 7
\]

\[
6.3^2 = 39.69 \quad 6.4^2 = 40.96
\]

The approximate value of \( \sqrt{40} \) is 6.3.

Simplifying Square Roots

To simplify a square root, extract any perfect squares within the expression.

**Example**

Simplify \( \sqrt{40} \).

\[
\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = 2\sqrt{10}
\]

Extracting Square Roots to Solve Equations

The solution to an equation where one term contains a variable and a constant term that are squared can be determined by extracting the square root. To do so, take the square root of both sides of the equation, and then isolate the variable to determine the value of the variable.

**Example**

\[
(x - 7)^2 = 75
\]

\[
\sqrt{(x - 7)^2} = \sqrt{75}
\]

\[
x - 7 = \pm\sqrt{75}
\]

\[
x = 7 \pm \sqrt{75}
\]

\[
x \approx 7 \pm 8.7
\]

\[
x \approx 15.7 \quad \text{or} \quad -1.7
\]
For a quadratic function that has zeros but cannot be factored, there exists another method for calculating the zeros of the function or solving the quadratic equation. Completing the square is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

When a function is written in standard form, \(ax^2 + bx + c\), the axis of symmetry is \(x = \frac{-b}{2a}\).

**Example**

Function: \(f(x) = x^2 + 4x + 1\)

\[x^2 + 4x + 1 = 0\]
\[x^2 + 4x = -1\]
\[x^2 + 4x + 4 = -1 + 4\]
\[x^2 + 4x + 4 = 3\]
\[(x + 2)^2 = 3\]
\[\sqrt{(x + 2)^2} = \pm\sqrt{3}\]

Check:

\[x + 2 = \pm\sqrt{3}\]

\[x = -2 \pm \sqrt{3}\]

\[x = \approx -2.268\]

\[(-2.268)^2 + 4(-2.268) + 1 = 0\]

\[x = 3.732\]

\[(-3.732)^2 + 4(-3.732) + 1 = 0\]

The roots are \(-2 \pm \sqrt{3}\).

The axis of symmetry is \(x = \frac{-4}{2}\), or \(x = -2\).